Basics of Deductive Program Verification

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Cours MPRI 2-36-1 “Preuve de Programme”

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Outline

Introduction, Short History
- Preliminary on Automated Deduction
  - Classical Propositional Logic
  - First-order logic
  - Logic Theories
  - Limitations of Automatic Provers

Introduction to Deductive Verification
- Formal contracts
- Hoare Logic
- Dijkstra’s Weakest Preconditions

“Modern” Approach, Blocking Semantics
- A ML-like Programming Language
- Blocking Operational Semantics
- Weakest Preconditions Revisited

Exercises

Preliminaries

Very first question
Lectures in English or in French?

» Schedule on the Web page https://marche.gitlabpages.inria.fr/lecture-deductive-verif/
» Lectures 1,2,3,4: Claude Marché
» Lectures 5,6,7,8: Jean-Marie Madiot
» Evaluation:
  » project \( P \) using the Why3 tool (http://why3.lri.fr)
  » final exam \( E \): date to decide
  » final mark = if \( P \geq E \) then \((E + P)/2\) else \((3E + P)/4\)
» Project:
  » provided at the beginning of January
  » due date around mid-February
» Internships (stages)

General Objectives

Ultimate Goal
Verify that software is free of bugs

Famous software failures:
http://www.cs.tau.ac.il/~nachumd/horror.html

This lecture
Computer-assisted approaches for verifying that a software conforms to a specification
Some general approaches to Verification

Static analysis, Algorithmic Verification
▶ model checking (automata-based models)
▶ abstract interpretation (domain-specific model, e.g. numerical)

Deductive verification
▶ formal models using expressive logics
▶ verification = computer-assisted mathematical proof

Refinement
▶ Formal models
▶ Code derived from model, correct by construction

A long time before success

Computer-assisted verification is an old idea
▶ Turing, 1948
▶ Floyd-Hoare logic, 1969

Success in practice: only from the mid-1990s
▶ Importance of the increase of performance of computers

A first success story:
▶ Paris metro line 14, using Atelier B (1998, refinement approach)

Other Famous Success Stories
http://www.astree.ens.fr/
▶ Microsoft’s hypervisor: using Microsoft’s VCC and the Z3 automated prover (2008, deductive verification)
More recently: verification of PikeOS
▶ Certified C compiler, developed using the Coq proof assistant (2009, correct-by-construction code generated by a proof assistant)
http://compcert.inria.fr/
▶ L4.verified micro-kernel, using tools on top of Isabelle/HOL proof assistant (2010, Haskell prototype, C code, proof assistant)
Other Success Stories at Industry

- Frama-C
  - EDF: abstract interpretation
  - Airbus: deductive verification
- Spark/Ada: Verification of Ada programs

https://www.adacore.com/industries

Remark
The two above use Why3 internally

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Exercises

Proposition logic in a nutshell

- Syntax:

  \[ \varphi ::= \bot \mid T \mid A, B \quad \text{(atoms)} \]
  \[ \mid \varphi \land \varphi \mid \varphi \lor \varphi \mid \neg \varphi \]
  \[ \mid \varphi \rightarrow \varphi \mid \varphi \leftrightarrow \varphi \]

- Semantics, models: truth tables

  \( \varphi \) is satisfiable if it has a model
  \( \varphi \) is valid if true in all models
  (equivalently \( \neg \varphi \) is not satisfiable)

SAT is \textit{decidable} \( \Leftrightarrow \) SAT solvers

Demo with Why3

\$ \texttt{why3 ide propositional.mlw} \$

Notice that Why3 indeed queries solvers for satisfiability of \( \neg \varphi \)

Focus on the “Tools” menu of Why3

File Edit Tools View Help

Task: propositional.mlw

File: propositional.mlw

Status: Theorems/Guards

Time: 0.00 seconds

Tools

- Alt-Ergo 2.3.1
- Coq 8.11.0
- CVC4 1.7
- Eprover 2.0
- Z3 6.8.6
- Auto level 0
- Auto level 1
- Auto level 2
- Auto level 3
- Split VC
- Edit
- Get Counterexamples
- Deploy valid absolute proofs
- Deploy all absolute proofs
- Clean node
- Remove node
- Interrupt

Support
First-order logic in a nutshell

▶ Syntax:

\[ \varphi ::= \ldots \]

\[ P(t, \ldots, t) \quad \text{(predicates)} \]

\[ \forall x. \varphi \quad \exists x. \varphi \]

\[ t ::= x \quad \text{variables} \]

\[ f(t, \ldots, t) \quad \text{(function symbols)} \]

▶ Semantics: models must interpret variables

▶ Satisfiability undecidable, but still semi-decidable: there exists complete systems of deduction rules (sequent calculus, natural deduction, superposition calculus)

▶ Examples of solvers: E, Spass, Vampire

  Implement refutationally complete procedure: if they answer 'unsat' then formula is unsatisfiable

Demo with Why3

`first-order.mlw`

Notice that Why3 logic is typed, and application is curried

Logic Theories

▶ Theory = set of formulas (called theorems) closed by logical consequence

▶ Axiomatic Theory = set of formulas generated by axioms (or axiom schemas)

▶ Consistent Theory

  for no \( P, P \) and \( \neg P \) are both theorems

  equivalently: 'false' is not a theorem

  equivalently: the theory has models

▶ Consistent Axiomatization

  'false' is not derivable

Theory of Equality

\[ \forall x. x = x \]

\[ \forall x, y. x = y \rightarrow y = x \]

\[ \forall x, y, z. x = y \land y = z \rightarrow x = z \]

(congruence) for all function symbols \( f \) of arity \( k \):

\[ \forall x_1, y_1, \ldots, x_k, y_k. x_1 = y_1 \land \cdots \land x_k = y_k \rightarrow f(x_1, \ldots, x_k) = f(y_1, \ldots, y_k) \]

and for all predicates \( p \) of arity \( k \):

\[ \forall x_1, y_1, \ldots, x_k, y_k. x_1 = y_1 \land \cdots \land x_k = y_k \rightarrow p(x_1, \ldots, x_k) \rightarrow p(y_1, \ldots, y_k) \]

Theory of Equality, Continued

\[ \forall x. x = x \]

\[ \forall x, y. x = y \rightarrow y = x \]

\[ \forall x, y, z. x = y \land y = z \rightarrow x = z \]

(congruence) \ldots

▶ General first-order deduction bad in such a case \( \rightarrow \)

  dedicated methods

  ▶ paramodulation

  ▶ congruence closure (for quantifier-free fragment)

▶ SMT solvers (Alt-Ergo, CVC4, Z3) implement dedicated (semi-)decision procedures

Demo with Why3

`equality.mlw`
Theories Continued

Theory of a given model
= formulas true in this model

- Central example: theory of linear integer arithmetic, i.e. formulas using $+$ and $\leq$
  - First-order theory is known to be decidable (Presburger)
  - SMT solvers typically implement a procedure for the existential fragment
- Also: theory of (non-linear) real arithmetic is decidable (Tarski)

Non-linear Integer Arithmetic

(a.k.a. Peano Arithmetic)

First-Order Integer Arithmetic
All valid first-order formulas on integers with $+$, $\times$ and $\leq$

- This theory is not even semi-decidable
- SMT solvers implement incomplete satisfiability checks: if solver answers 'unsat' then it is unsatisfiable

Demo with Why3
arith.mlw

Digression about Non-linear Integer Arithmetic

Representation Theorem (Gödel)
Every recursive function $f$ is representable by a predicate $\varphi_f$ such that

$$\varphi_f(x_1, \ldots, x_k, y)$$

is true if and only if

$$y = f(x_1, \ldots, x_k)$$

First incompleteness Theorem (Gödel)
That theory is not recursively axiomatizable

Summary of prover limitations

- Superposition solvers (E, Spass, )
  - do not support well theories except equality
  - do quite well with quantifiers
- SMT solvers (Alt-Ergo, CVC4, Z3)
  - several theories are built-in
  - weaker with quantifiers
- None support reasoning by induction
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Exercises

IMP language

A very basic imperative programming language
  • only global variables
  • only integer values for expressions
  • basic statements:
    ▶ assignment \( x \leftarrow e \)
    ▶ sequence \( S_1; S_2 \)
    ▶ conditionals if \( e \) then \( S_1 \) else \( S_2 \)
    ▶ loops while \( e \) do \( S \)
    ▶ no-op skip

Formal Contracts

General form of a program:

\[
\text{Contract}
\]

\begin{itemize}
  \item \textbf{precondition}: expresses what is assumed before running
          the program
  \item \textbf{post-condition}: expresses what is supposed to hold when
          program exits
\end{itemize}

Demo with Why3

contracts.mlw

Hoare triples

\begin{itemize}
  \item \textbf{Hoare triple} : notation \( \{P\} s \{Q\} \)
  \item \( P \) : formula called the \textbf{precondition}
  \item \( Q \) : formula called the \textbf{postcondition}
\end{itemize}

\[
\text{Intended meaning}
\]

\( \{P\} s \{Q\} \) is true if and only if:
when the program \( s \) is executed in any state satisfying \( P \), then
(if execution terminates) its resulting state satisfies \( Q \)

This is a \textbf{Partial Correctness}: we say nothing if \( s \) does not terminate
Examples

Examples of valid triples for partial correctness:

- \{x = 1\} x \leftarrow x + 2\{x = 3\}
- \{x = y\} x \leftarrow x + y\{x = 2 + y\}
- \{\exists v. x = 4 \times v\} x \leftarrow x + 42\{\exists w. x = 2 \times w\}
- \{true\} while 1 do skip{false}

Running Example

Three global variables \(n\), \(count\), and \(sum\)

count \leftarrow 0; sum \leftarrow 1;
while \(sum \leq n\) do
    count \leftarrow count + 1; sum \leftarrow sum + 2 \times count + 1

What does this program compute?
(assuming input is \(n\) and output is \(count\))

Informal specification:
- at the end of execution of this program, \(count\) contains the square root of \(n\), rounded downward
- e.g. for \(n = 42\), the final value of \(count\) is 6.

See file imp_isqrt.mlw

Hoare logic as an Axiomatic Semantics

Original Hoare logic \([\sim 1970]\)
Axiomatic Semantics of programs

Set of inference rules producing triples

\[
\begin{align*}
\{P\} \text{skip}\{P\} \\
\{P[x \leftarrow e]\} x \leftarrow e\{P\} \\
\{P\} s_1\{Q\} \quad \{Q\} s_2\{R\} \\
\{P\} s_1; s_2\{R\}
\end{align*}
\]

- Notation \(P[x \leftarrow e]\): replace all occurrences of program variable \(x\) by \(e\) in \(P\).

Hoare Logic, continued

Frame rule:

\[
\frac{\{P\} s\{Q\}}{\{P \land R\} s\{Q \land R\}}
\]

with \(R\) a formula where no variables assigned in \(s\) occur

Consequence rule:

\[
\frac{\{P\} s\{Q'\} \vdash P \rightarrow P'}{\{P\} s\{Q\}} \quad \frac{Q' \rightarrow Q}{\{P\} s\{Q\}}
\]

- Example: proof of

\[
\{x = 1\} x \leftarrow x + 2\{x = 3\}
\]
Proof of the example

\[
\begin{align*}
(x + 2 = 3) & \iff x = 1 \rightarrow x + 2 = 3 \\
(x + 2 = 3) & \iff x = 3 \rightarrow x = 3 \\
\{x = 1\} x < x + 2 & \{x = 3\} \\
\{x = 1\} x < x + 2 & \{x = 3\}
\end{align*}
\]

Hoare Logic, continued

Rules for if and while:

\[
\begin{align*}
\{P \land e\} s_1 & \{Q\} & \{P \land \lnot e\} s_2 & \{Q\} \\
\{P\} & \text{if } e \text{ then } s_1 \text{ else } s_2 & \{Q\} \\
\{I \land e\} s & \{I\} & \{I\} \text{ while } e \text{ do } s & \{I \land \lnot e\}
\end{align*}
\]

\(I\) is called a loop invariant

Informal justification of the while rule

\[
\begin{align*}
\{I \land e\} s & \{I\} \\
\{I\} \text{ while } e \text{ do } s & \{I \land \lnot e\}
\end{align*}
\]

Exercise: prove of the triple

\(\{n \geq 0\} \ ISQR\ \{\text{count}^2 \leq n \land n < (\text{count} + 1)^2\}\)

Could we do that by hand?

Back to demo: file \texttt{imp_isqrt.mlw}

Example: isqrt(42)

Warning

Finding an adequate loop invariant is a major difficulty
Operational semantics

[Plotkin 1981, structural operational semantics (SOS)]

- we use a standard small-step semantics
- program state: describes content of global variables at a given time. It is a finite map \( \Sigma \) associating to each variable \( x \) its current value denoted \( \Sigma(x) \).
- Value of an expression \( e \) in some state \( \Sigma \):
  - denoted \( [e]_{\Sigma} \)
  - always defined, by the following recursive equations:
    \[
    [n]_{\Sigma} = n \\
    [x]_{\Sigma} = \Sigma(x) \\
    [e_1 \text{ op } e_2]_{\Sigma} = [e_1]_{\Sigma} \text{ op } [e_2]_{\Sigma}
    \]
- \( [\text{op}] \) natural semantic of operator \( \text{op} \) on integers (with relational operators returning 0 for false and \( \neq 0 \) for true).

Semantics of statements

Semantics of statements: defined by judgment

\[
\Sigma, s \leadsto \Sigma', s'
\]

meaning: in state \( \Sigma \), executing one step of statement \( s \) leads to the state \( \Sigma' \) and the remaining statement to execute is \( s' \).

The semantics is defined by the following rules.

\[
\Sigma, x \leftarrow e \leadsto \Sigma \{ x \leftarrow [e]_{\Sigma} \}, \text{skip}
\]

\[
\Sigma, s_1 \leadsto \Sigma', s'_1 \\
\Sigma, (s_1; s_2) \leadsto \Sigma', (s'_1; s_2)
\]

\[
\Sigma, (\text{skip}; s) \leadsto \Sigma, s
\]
Execution of programs

▶ ⇝: a binary relation over pairs (state,statement)
▶ transitive closure: ⇝+
▶ reflexive-transitive closure: ⇝∗

In other words:

\[ \Sigma, s \xrightarrow{*} \Sigma', s' \]

means that statement \( s \), in state \( \Sigma \), reaches state \( \Sigma' \) with remaining statement \( s' \) after executing some finite number of steps.

Running example:

\{n = 42, count =?, sum =?\}, ISQR \xrightarrow{*}
\{n = 42, count = 6, sum = 49\}, skip

Execution and termination

▶ any statement except skip can execute in any state
▶ the statement skip alone represents the final step of execution of a program
▶ there is no possible runtime error.

Definition

Execution of statement \( s \) in state \( \Sigma \) terminates if there is a state \( \Sigma' \) such that \( \Sigma, s \xrightarrow{*} \Sigma', \text{skip} \)

▶ since there are no possible runtime errors, \( s \) does not terminate means that \( s \) diverges (i.e. executes infinitely).

Semantics of formulas

▶ \([p]_{\Sigma,V}\) denotes the semantics of formula \( p \) in program state \( \Sigma \) and mapping \( V \) of logic variables to integers
▶ defined recursively, e.g.

\[
\begin{align*}
[p_1 \land p_2]_{\Sigma,V} &= \begin{cases} 
\top & \text{if } [p_1]_{\Sigma,V} = \top \text{ and } [p_2]_{\Sigma,V} = \top \\
\bot & \text{otherwise}
\end{cases} \\
[\forall v.e]_{\Sigma,V} &= \top \text{ if for all } n. \ [e]_{\Sigma,V[v \leftarrow n]} = \top \\
[v]_{\Sigma,V} &= V(v) \\
[x]_{\Sigma,V} &= \Sigma(x)
\end{align*}
\]

Notations:

▶ \( \Sigma \models p \): the formula \( p \) is valid in \( \Sigma \) i.e. \([p]_{\Sigma,\emptyset}\) is \( \top \)
▶ \( \models p \): formula \([p]_{\Sigma,\emptyset}\) holds in all states \( \Sigma \).

Soundness

Definition (Partial correctness)

Hoare triple \( \{P\} s \{Q\} \) is said valid if:

for any states \( \Sigma, \Sigma' \), if

▶ \( \Sigma, s \xrightarrow{*} \Sigma', \text{skip} \) and
▶ \( \Sigma \models P \)
then \( \Sigma' \models Q \)

Theorem (Soundness of Hoare logic)

The set of rules is correct: any derivable triple is valid.

This is proved by induction on the derivation tree of the considered triple.

For each rule: assuming that the triples in premises are valid, we show that the triple in conclusion is valid too.
Digression: Completeness of Hoare Logic

Two major difficulties for proving a program

- *guess the appropriate intermediate formulas* (for sequence, for the loop invariant)
- *prove the logical premises of consequence rule*

Theoretical question: completeness. Are all valid triples derivable from the rules?

**Theorem (Relative Completeness of Hoare logic)**

The set of rules of Hoare logic is *relatively* complete: if the logic language is *expressive enough*, then any valid triple \( \{P\} s \{Q\} \) can be derived using the rules.


Yet, this does not provide an effective recipe to discover suitable loop invariants (see also the theory of abstract interpretation [Cousot, 1990])

### Weakest liberal precondition

[Dijkstra 1975]

Function **WLP\((s, Q)\)**:

- \(s\) is a statement
- \(Q\) is a formula
- returns a formula

It should return the *minimal precondition* \(P\) that validates the triple \(\{P\} s \{Q\}\)

### Definition of **WLP\((s, Q)\)**

Recursive definition:

\[
\begin{align*}
\text{WLP}(\text{skip}, Q) & = Q \\
\text{WLP}(x \leftarrow e, Q) & = Q[x \leftarrow e] \\
\text{WLP}(s_1; s_2, Q) & = \text{WLP}(s_1, \text{WLP}(s_2, Q)) \\
\text{WLP}(\text{if } e \text{ then } s_1 \text{ else } s_2, Q) & = (e \rightarrow \text{WLP}(s_1, Q)) \land (\neg e \rightarrow \text{WLP}(s_2, Q))
\end{align*}
\]
Definition of $WLP(s, Q)$, continued

$$WLP(\text{while } e \text{ invariant } I \text{ do } s, Q) =$$

$$I \land$$

$$(\forall v_1, \ldots, v_k.$$)

$$((e \land I) \rightarrow WLP(s, I))$$

$$(\text{invariant preserved})$$

$$\land((\neg e \land I) \rightarrow Q))$$

$$(\text{invariant implies post})$$

$[w_i \leftarrow v_i]$$

$$(\text{invariant implies post})$$

where $w_1, \ldots, w_k$ is the set of assigned variables in statement $s$

and $v_1, \ldots, v_k$ are fresh logic variables

Examples

$$WLP(x \leftarrow x + y, x = 2y) \equiv x + y = 2y$$

$$WLP(\text{while } y > 0 \text{ invariant even}(y) \text{ do } y \leftarrow y - 2, even(y)) \equiv$$

$$\forall v, ((v > 0 \land even(v)) \rightarrow even(v - 2))$$

$$\land((v \leq 0 \land even(v)) \rightarrow even(v))$$

Soundness

**Theorem (Soundness)**

For all statement $s$ and formula $Q$, $\{WLP(s, Q)\} s \{Q\}$ is valid.

Proof by induction on the structure of statement $s$.

Consequence

For proving that a triple $\{P\} s \{Q\}$ is valid, it suffices to prove the formula $P \rightarrow WLP(s, Q)$.

This is basically the goal that Why3 produces

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- Exercises
Beyond IMP and classical Hoare Logic

Extended language

- more data types
- *logic variables*: local and immutable
- *labels* in specifications

Handle termination issues:

- prove properties on non-terminating programs
- prove termination when wanted

Prepare for adding later:

- run-time errors (how to prove their absence)
- local *mutable* variables, functions
- complex data types

Extended Syntax: Generalities

- We want a few basic data types: int, bool, real, unit
- *No difference between expressions and statements* anymore

Basically we consider

- A purely functional language (ML-like)
- with *global mutable variables*

  *very restricted notion of modification of program states*

Base Data Types, Operators, Terms

- unit type: type `unit`, only one constant (`()`)
- Booleans: type `bool`, constants `True, False`, operators `and, or, not`
- integers: type `int`, operators `+, -, x` (no division)
- reals: type `real`, operators `+, -, x` (no division)
- Comparisons of integers or reals, returning a boolean
- “if-expression”: written `if b then t_1 else t_2`

```
 t ::= val (values, i.e. constants)  
   | v (logic variables)  
   | x (program variables)  
   | t op t (binary operations)  
   | if t then t else t (if-expression)
```

Local logic variables

We extend the syntax of terms by

```
t ::= let v = t in t
```

Example: approximated cosine

```
let cos_x =  
  let y = x*x in  
  1.0 - 0.5 * y + 0.0416666 * y * y  
in  
...```
Practical Notes

Theorem provers (inc. Alt-Ergo, CVC4, Z3) typically support such a typed logic
▶ may also support if-expressions and let bindings

Alternatively, Why3 manages to transform terms and formulas when needed (e.g. transformation of if-expressions and/or let-expressions into equivalent formulas)

Syntax: Formulas

It is (typed) first-order logic, as in previous lecture, but also with addition of local binding:

```
p ::= t
  | p ∧ p | p ∨ p | ¬p | p → p  (connectives)
  | ∀ v : τ, p | ∃ v : τ, p  (quantification)
  | let v = t in p  (local binding)
```

Typing

▶ Types:

```
τ ::= int | real | bool | unit
```

▶ Typing judgment:

```
Γ ⊢ t : τ
```

where Γ maps identifiers to types:
▶ either v : τ (logic variable, immutable)
▶ either x : mut τ (program variable, mutable)

Important
▶ a mutable variable is not a value (it is not a “reference” value)
▶ as such, there is no “reference on a reference”
▶ no aliasing

Typing rules

Constants:

```
Γ ⊢ n : int
Γ ⊢ r : real
Γ ⊢ True : bool
Γ ⊢ False : bool
```

Variables:

```
Γ ⊢ v : τ ∈ Γ  \quad x : mut τ ∈ Γ
Γ ⊢ v : τ
Γ ⊢ x : τ
```

Let binding:

```
Γ ⊢ t₁ : τ₁
{v : τ₁} : Γ ⊢ t₂ : τ₂
Γ ⊢ let v = t₁ in t₂ : τ₂
```

▶ All terms have a base type (not a “reference”)
▶ In practice: Why3, unlike OCaml, does not require to write !x for mutable variables
Formal Semantics: Terms and Formulas

Program states are augmented with a stack of local (immutable) variables

- $\Sigma$: maps program variables to values (a map)
- $\pi$: maps logic variables to values (a stack)

$$
\begin{align*}
\llbracket \text{val} \rrbracket_{\Sigma,\pi} &= \text{val} & (\text{values}) \\
\llbracket x \rrbracket_{\Sigma,\pi} &= \Sigma(x) & \text{if } x : \text{mut } \tau \\
\llbracket v \rrbracket_{\Sigma,\pi} &= \pi(v) & \text{if } v : \tau \\
\llbracket t_1 \ op \ t_2 \rrbracket_{\Sigma,\pi} &= \llbracket t_1 \rrbracket_{\Sigma,\pi} \ op \ \llbracket t_2 \rrbracket_{\Sigma,\pi} \\
\llbracket \text{let } v = t_1 \ \text{in} \ t_2 \rrbracket_{\Sigma,\pi} &= \llbracket t_2 \rrbracket_{\Sigma,\{v=\llbracket t_1 \rrbracket_{\Sigma,\pi}\} \cdot \pi}
\end{align*}
$$

Warning

Semantics is a partial function, it is not defined on ill-typed formulas

Common notation for formulas

$\Sigma, \pi \models \varphi$ means $\llbracket \varphi \rrbracket_{\Sigma,\pi} = \text{true}$

Expressions: generalities

- Former statements of IMP are now expressions of type unit
- Expressions may have Side Effects
- Statement `skip` is identified with `()`
- The sequence is replaced by a local binding
- From now on, the condition of the `if then else` and the `while do` in programs is a Boolean expression

Syntax

```latex
\begin{align*}
\mathit{e} & ::= \quad \mathit{t} & \text{(pure term)} \\
& | \quad \mathit{e} \ \mathit{op} \ \mathit{e} & \text{(binary operation)} \\
& | \quad x \leftarrow \mathit{e} & \text{(assignment)} \\
& | \quad \text{let } v = \mathit{e} \ \text{in} \ \mathit{e} & \text{(local binding, immutable)} \\
& | \quad \text{if } \mathit{e} \ \text{then } \mathit{e} \ \text{else } \mathit{e} & \text{(conditional)} \\
& | \quad \text{while } \mathit{e} \ \text{do } \mathit{e} & \text{(loop)} \\
\end{align*}
```

- `sequence $e_1; e_2$` : syntactic sugar for
  ```latex
  \begin{align*}
  \text{let } v = e_1 \ \text{in} \ \ e_2
  \end{align*}
  ```
  when $e_1$ has type `unit` and $v$ not used in $e_2$

Type Soundness Property

Our logic language satisfies the following standard property of purely functional language

**Theorem (Type soundness)**

Every well-typed terms and well-typed formulas have a semantics

Proof: induction on the derivation tree of well-typing
Toy Examples

\[
\begin{align*}
z & \leftarrow \text{if } x \geq y \text{ then } x \text{ else } y \\
\text{let } v &= r \text{ in } (r \leftarrow v + 42; v) \\
\text{while } (x \leftarrow x - 1; x > 0) & \text{ do } () \\
\text{while } (\text{let } v = x \text{ in } x \leftarrow x - 1; v > 0) & \text{ do } ()
\end{align*}
\]

Typing Rules for Expressions

Assignment: \[
\frac{x : \text{mut } \tau \in \Gamma}{\Gamma \vdash x : \tau}
\frac{\Gamma \vdash e : \tau}{\Gamma \vdash x < - e : \text{unit}}
\]

Let binding: \[
\frac{\Gamma \vdash e_1 : \tau_1 \quad \{v : \tau_1\} \cdot \Gamma \vdash e_2 : \tau_2}{\Gamma \vdash \text{let } v = e_1 \text{ in } e_2 : \tau_2}
\]

Conditional: \[
\frac{\Gamma \vdash c : \text{bool} \quad \Gamma \vdash e_1 : \tau \quad \Gamma \vdash e_2 : \tau}{\Gamma \vdash \text{if } c \text{ then } e_1 \text{ else } e_2 : \tau}
\]

Loop: \[
\frac{\Gamma \vdash c : \text{bool} \quad \Gamma \vdash e : \text{unit}}{\Gamma \vdash \text{while } c \text{ do } e : \text{unit}}
\]

Operational Semantics

Novelty w.r.t. IMP

Need to precise the order of evaluation: left to right
(e.g. \( x < 0; ((x < 1) \cdot 2) + x \) = 2 or 3 ?)

- one-step execution has the form
  \[
  \Sigma, \pi, e \rightsquigarrow \Sigma', \pi', e'
  \]
  \( \pi \) is the stack of local variables
- values do not reduce

Operational Semantics

- Assignment
  \[
  \Sigma, \pi, e \rightsquigarrow \Sigma', \pi', e' \quad \Sigma, \pi, x < - e \rightsquigarrow \Sigma', \pi', x < - e'
  \]
  \[
  \Sigma, \pi, x < \text{val} \rightsquigarrow \Sigma[x \leftarrow \text{val}], \pi, ()
  \]

- Let binding
  \[
  \Sigma, \pi, e_1 \rightsquigarrow \Sigma', \pi', e'_1 \quad \Sigma, \pi, \text{let } v = e_1 \text{ in } e_2 \rightsquigarrow \Sigma', \pi', \text{let } v = e'_1 \text{ in } e_2
  \]
  \[
  \Sigma, \pi, \text{let } v = \text{val} \text{ in } e \rightsquigarrow \Sigma, \{v = \text{val}\} \cdot \pi, e
  \]
Operational Semantics, Continued

- **Binary operations**

  \[
  \begin{align*}
  &\Sigma, \pi, e_1 \leadsto \Sigma', \pi', e'_1 \\
  &\Sigma, \pi, e_1 + e_2 \leadsto \Sigma', \pi', e'_1 + e'_2 \\
  &\Sigma, \pi, e_2 \leadsto \Sigma', \pi', e'_2 \\
  &\Sigma, \pi, val_1 + e_2 \leadsto \Sigma', \pi', val_1 + e'_2 \\
  &val = val_1 + val_2 \\
  &\Sigma, \pi, val_1 + val_2 \leadsto \Sigma, \pi, val
  \end{align*}
  \]

Context Rules versus Let Binding

Remark: most of the context rules can be avoided

- An equivalent operational semantics can be defined using `let v = ... in ...` instead, e.g.:

  \[
  \begin{align*}
  &v_1, v_2 \text{ fresh} \\
  &\Sigma, \pi, e_1 + e_2 \leadsto \Sigma, \pi, \text{let } v_1 = e_1 \text{ in let } v_2 = e_2 \text{ in } v_1 + v_2 \\
  \end{align*}
  \]

  Thus, only the context rule for let is needed

Operational Semantics, Continued

- **Conditional**

  \[
  \begin{align*}
  &\Sigma, \pi, c \leadsto \Sigma', \pi', c' \\
  &\Sigma, \pi, \text{if } c \text{ then } e_1 \text{ else } e_2 \leadsto \Sigma', \pi', \text{if } c' \text{ then } e_1 \text{ else } e_2 \\
  &\Sigma, \pi, \text{if } True \text{ then } e_1 \text{ else } e_2 \leadsto \Sigma, \pi, e_1 \\
  &\Sigma, \pi, \text{if } False \text{ then } e_1 \text{ else } e_2 \leadsto \Sigma, \pi, e_2 \\
  \end{align*}
  \]

- **Loop**

  \[
  \begin{align*}
  &\Sigma, \pi, \text{while } c \text{ do } e \leadsto \\
  &\Sigma, \pi, \text{if } c \text{ then } (e; \text{while } c \text{ do } e) \text{ else } ()
  \end{align*}
  \]

Type Soundness

**Theorem**

Every well-typed expression evaluates to a value or executes infinitely.

Classical proof:

- type is preserved by reduction
- execution of well-typed expressions that are not values can progress
**Blocking Semantics: General Ideas**

- add **assertions** in expressions
- failed assertions = "run-time errors"

First step: modify expression syntax with
- new expression: assertion
- adding loop invariant in loops

\[
e ::= \text{assert } p \quad \text{(assertion)}
\|
\text{while } e \text{ invariant } / \text{ do } e \quad \text{(annotated loop)}
\]

**Toy Examples**

```plaintext
z <- if x >= y then x else y ;
assert (z >= x \&\& z >= y)
```

```plaintext
while (x <- x - 1; x > 0)
(* (--x > 0) in C *)
invariant x >= 0 do ()
assert (x = 0)
```

```plaintext
while (let v = x in x <- x - 1; v > 0)
(* (x-- > 0) in C *)
invariant x >= -1 do ()
assert (x = -1)
```

**Blocking Semantics: Modified Rules**

\[
\frac{\llbracket P \rrbracket_{\Sigma, \pi} \text{ holds}}{\Sigma, \pi, \text{assert } P \leadsto \Sigma, \pi, ()}
\]

\[
\frac{\llbracket I \rrbracket_{\Sigma, \pi} \text{ holds}}{\Sigma, \pi, \text{while } c \text{ invariant } / \text{ do } e \leadsto
\Sigma, \pi, \text{if } c \text{ then } (e; \text{while } c \text{ invariant } / \text{ do } e) \text{ else } ()}
\]

**Important remark**
Execution blocks as soon as an invalid annotation is met

**Definition (Safety of execution)**
Execution of an expression in a given state is **safe** if it does not block: either terminates on a value or runs infinitely.

**Hoare triples: result value in post-conditions**

New addition in the logic language:
- keyword **result** in post-conditions
- denotes the value of the expression executed

Example:

\[
\{ \text{true } \}
\quad \text{if } x >= y \text{ then } x \text{ else } y
\]

\[
\{ \text{result } >= x \&\& \text{result } >= y \}
\]
**Hoare triples: Soundness**

**Definition (validity of a triple)**
A triple \( \{P\} e \{Q\} \) is valid if for any state \( \Sigma, \pi \) satisfying \( P \), \( e \) executes safely in \( \Sigma, \pi \), and if it terminates, the final state satisfies \( Q \).

**Difference with historical Hoare triples**
Validity of a triple now implies safety of its execution, even if it does not terminate.

---

**Weakest Preconditions Revisited**

**Goal:**
- construct a new calculus \( \text{WP}(e, Q) \)

**Expected property:** in any state satisfying \( \text{WP}(e, Q) \),
- \( e \) is guaranteed to execute safely
- if it terminates, \( Q \) holds in the final state

**Difference with historical WLP calculus**
This calculus is no more “liberal”, the computed precondition guarantees safety of execution, even if it does not terminate.

---

**New Weakest Precondition Calculus**

**Pure expressions (i.e. without side-effects, a.k.a. “terms”)**
\[
\text{WP}(t, Q) = Q[\text{result} \leftarrow t]
\]

**‘let’ binding**
\[
\text{WP}(\text{let } x = e_1 \text{ in } e_2, Q) = \\
\text{WP}(e_1, (\text{WP}(e_2, Q)[x \leftarrow \text{result}]))
\]

Reminder: sequence is a particular case of ‘let’
\[
\text{WP}((e_1; e_2), Q) = \text{WP}(e_1, \text{WP}(e_2, Q))
\]

---

**Weakest Preconditions, continued**

**Assignment:**
\[
\text{WP}(x \leftarrow e, Q) = \text{WP}(e, Q[\text{result} \leftarrow (); x \leftarrow \text{result}])
\]

**Alternative:**
\[
\text{WP}(x \leftarrow e, Q) = \text{WP}(\text{let } v = e \text{ in } x \leftarrow v, Q) \\
\text{WP}(x \leftarrow t, Q) = Q[\text{result} \leftarrow (); x \leftarrow t]
\]
Weakest Preconditions, continued

- Conditional
  \[ \text{WP}(\text{if } e_1 \text{ then } e_2 \text{ else } e_3, Q) = \text{WP}(e_1, \text{if } \text{result} \text{ then } \text{WP}(e_2, Q) \text{ else } \text{WP}(e_3, Q)) \]

- Alternative with let: (exercise!)

Soundness of WP

**Lemma (Preservation by Reduction)**

If \( \Sigma, \pi \models \text{WP}(e, Q) \) and \( \Sigma, \pi, e \rightsquigarrow \Sigma', \pi', e' \) then \( \Sigma', \pi' \models \text{WP}(e', Q) \)

Proof: predicate induction of \( \rightsquigarrow \).

**Lemma (Progress)**

If \( \Sigma, \pi \models \text{WP}(e, Q) \) and \( e \) is not a value then there exists \( \Sigma', \pi, e' \) such that \( \Sigma, \pi, e \rightsquigarrow \Sigma', \pi', e' \)

Proof: structural induction of \( e \).

**Corollary (Soundness)**

If \( \Sigma, \pi \models \text{WP}(e, Q) \) then

- \( e \) executes safely in \( \Sigma, \pi \).
- if execution terminates, \( Q \) holds in the final state.
Exercise 1

Consider the following (inefficient) program for computing the sum \(a + b\).

\[
x \leftarrow a; \ y \leftarrow b;
\]
\[
\text{while } y > 0 \text{ do}
\]
\[
x \leftarrow x + 1; \ y \leftarrow y - 1
\]

(Why3 file to fill in: imp_sum.mlw)

- Propose a post-condition stating that the final value of \(x\) is the sum of the values of \(a\) and \(b\)
- Find an appropriate loop invariant
- Prove the program.

Exercise 2

The following program is one of the original examples of Floyd.

\[
q \leftarrow 0; \ r \leftarrow x;
\]
\[
\text{while } r \geq y \text{ do}
\]
\[
r \leftarrow r - y; \ q \leftarrow q + 1
\]

(Why3 file to fill in: imp_euclidean_div.mlw)

- Propose a formal precondition to express that \(x\) is assumed non-negative, \(y\) is assumed positive, and a formal post-condition expressing that \(q\) and \(r\) are respectively the quotient and the remainder of the Euclidean division of \(x\) by \(y\).
- Find appropriate loop invariants and prove the correctness of the program.

Exercise 3

Let's assume given in the underlying logic the functions \(\text{div2}(x)\) and \(\text{mod2}(x)\) which respectively return the division of \(x\) by 2 and its remainder. The following program is supposed to compute, in variable \(r\), the power \(x^n\).

\[
r \leftarrow 1; \ p \leftarrow x; \ e \leftarrow n;
\]
\[
\text{while } e > 0 \text{ do}
\]
\[
\quad \text{if } \text{mod2}(e) \neq 0 \text{ then } r \leftarrow r \times p;
\]
\[
\quad p \leftarrow p \times p;
\]
\[
\quad e \leftarrow \text{div2}(e);
\]

(Why3 file to fill in: power_int.mlw)

- Assuming that the power function exists in the logic, specify appropriate pre- and post-conditions for this program.
- Find an appropriate loop invariant, and prove the program.
Exercise 4

The Fibonacci sequence is defined recursively by \( fib(0) = 0 \), \( fib(1) = 1 \) and \( fib(n + 2) = fib(n + 1) + fib(n) \). The following program is supposed to compute \( fib \) in linear time, the result being stored in \( y \).

\[
y <- 0; x <- 1; i <- 0;
while i < n do
    aux <- y; y <- x; x <- x + aux; i <- i + 1
\]

▶ Assuming \( fib \) exists in the logic, specify appropriate pre- and post-conditions.
▶ Prove the program.

Exercise (original Floyd rule for assignment)

1. Prove that the triple

\[
\{ P \} x <- e[\exists v, \ e[x <- v] = x \land P[x <- v]]
\]

is valid with respect to the operational semantics.

2. Show that the triple above can be proved using the rules of Hoare logic.

Let us assume that we replace the standard Hoare rule for assignment by the Floyd rule

\[
\{ P \} x <- e[\exists v, \ e[x <- v] = x \land P[x <- v]]
\]

3. Show that the triple \( \{ P[x <- e] \} x <- e(P) \) can be proved with the new set of rules.

Bibliography


