Basics of Deductive Program Verification

Claude Marché

Cours MPRI 2-36-1 “Preuve de Programme”

9 décembre 2020

Outline

Introduction, Short History

Preliminary on Automated Deduction
  Classical Propositional Logic
  First-order logic
  Logic Theories
  Limitations of Automatic Provers

Introduction to Deductive Verification
  Formal contracts
  Hoare Logic
  Dijkstra’s Weakest Preconditions

Exercises

Preliminaries

Very first question

Lectures in English or in French?

▶ Schedule on the Web page https://marche.gitlabpages.inria.fr/lecture-deductive-verif/
▶ Lectures 1,2,3,4: Claude Marché
▶ Lectures 5,6,7,8: Jean-Marie Madiot
▶ (to confirm) some lecture could be replaced by practical lab, support for project. Project due date around mid-February.

Evaluation:

▶ project $P$ using the Why3 tool (http://why3.lri.fr)
▶ final exam $E$: date to decide
▶ final mark $= (2E + P + \max(E,P))/4$
▶ internships (stages)

General Objectives

Ultimate Goal

Verify that software is free of bugs

Famous software failures:

http://www.cs.tau.ac.il/~nachumd/horror.html

This lecture

Computer-assisted approaches for verifying that a software conforms to a specification
Some general approaches to Verification

Static analysis, Algorithmic Verification
- **model checking** (automata-based models)
- **abstract interpretation** (domain-specific model, e.g. numerical)

Deductive verification
- formal models using expressive logics
- verification = computer-assisted mathematical proof

A long time before success

Computer-assisted verification is an old idea
- **Turing**, 1948
- **Floyd-Hoare logic**, 1969

Success in practice: only from the mid-1990s
- Importance of the *increase of performance of computers*

A first success story:
- **Paris metro line 14**, using **Atelier B** (1998, refinement approach)

Other Famous Success Stories

- **Flight control software of A380**: Astree verifies absence of run-time errors (2005, abstract interpretation)
- **Microsoft's hypervisor**: using Microsoft's VCC and the Z3 automated prover (2008, deductive verification)
- More recently: verification of PikeOS
- **Certified C compiler**, developed using the Coq proof assistant (2009, correct-by-construction code generated by a proof assistant)
  [http://compcert.inria.fr/](http://compcert.inria.fr/)
- **L4.verified micro-kernel**, using tools on top of Isabelle/HOL proof assistant (2010, Haskell prototype, C code, proof assistant)
Other Success Stories at Industry

- Frama-C
- EDF: abstract interpretation
- Airbus: deductive verification
- Spark/Ada: Verification of Ada programs
  https://www.adacore.com/industries

Remark
The two above use Why3 internally

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Exercises

Proposition logic in a nutshell

- Syntax:
  \[ \varphi ::= \bot \mid T \mid A, B \quad \text{(atoms)} \]
  \[ \mid \varphi \land \varphi \mid \varphi \lor \varphi \mid \neg \varphi \]
  \[ \mid \varphi \rightarrow \varphi \mid \varphi \leftrightarrow \varphi \]

- Semantics, models: truth tables
  \( \varphi \) is satisfiable if it has a model
  \( \varphi \) is valid if true in all models
  (equivalently \( \neg \varphi \) is not satisfiable)

SAT is \textit{decidable} \iff SAT solvers

Demo with Why3

\$ why3 ide propositional.mlw

Notice that Why3 indeed queries solvers for satisfiability of \( \neg \varphi \)
First-order logic in a nutshell

▶ Syntax:

\[ \varphi ::= \ldots \mid P(t, \ldots, t) \quad \text{(predicates)} \]

\[ \mid \forall x. \varphi \mid \exists x. \varphi \]

\[ t ::= x \quad \text{variables} \]

\[ \mid f(t, \ldots, t) \quad \text{(function symbols)} \]

▶ Semantics: models must interpret variables. C

▶ Satisfiability **undecidable**, but still **semi-decidable**: there exists complete systems of deduction rules (sequent calculus, natural deduction, superposition calculus)

▶ Examples of solvers: E, Spass, Vampire

Implement **refutationally complete** procedure:

- if they answer ‘unsat’ then formula is unsatisfiable

Demo with Why3

`first-order.mlw`

Notice that Why3 logic is **typed**, and application is curryied

### Logic Theories

▶ **Theory** = set of formulas (called **theorems**) closed by logical consequence

▶ **Axiomatic Theory** = set of formulas generated by axioms (or axiom schemas)

▶ **Consistent Theory**

- for no \( P, P \) and \( \neg P \) are both theorems
- equivalently: ‘false’ is not a theorem
- equivalently: the theory has models

▶ **Consistent Axiomatization**

- ‘false’ is not derivable

### Theory of Equality

\[ \forall x. x = x \]

\[ \forall x, y. x = y \rightarrow y = x \]

\[ \forall x, y, z. x = y \land y = z \rightarrow x = z \]

(congruence) for all function symbols \( f \) of arity \( k \):

\[ \forall x_1, y_1, \ldots, x_k, y_k. x_1 = y_1 \land \cdots \land x_k = y_k \rightarrow f(x_1, \ldots, x_k) = f(y_1, \ldots, y_k) \]

and for all predicates \( p \) of arity \( k \):

\[ \forall x_1, y_1, \ldots, x_k, y_k. x_1 = y_1 \land \cdots \land x_k = y_k \rightarrow p(x_1, \ldots, x_k) \rightarrow p(y_1, \ldots, y_k) \]

### Theory of Equality, Continued

\[ \forall x. x = x \]

\[ \forall x, y. x = y \rightarrow y = x \]

\[ \forall x, y, z. x = y \land y = z \rightarrow x = z \]

(congruence) ... 

▶ General first-order deduction bad in such a case \( \rightarrow \) dedicated methods

- **paramodulation**
- **congruence closure** (for quantifier-free fragment)

▶ SMT solvers (Alt-Ergo, CVC4, Z3) implement dedicated (semi-)decision procedures

Demo with Why3

`equality.mlw`
Theories Continued

Theory of a given model
= formulas true in this model

- Central example: theory of linear integer arithmetic, i.e. formulas using + and ≤
  - First-order theory is known to be decidable (Presburger)
  - SMT solvers typically implement a procedure for the existential fragment
- Also: theory of (non-linear) real arithmetic is decidable (Tarski)

Digression about Non-linear Integer Arithmetic

Representation Theorem (Gödel)
Every recursive function \( f \) is representable by a predicate \( \phi_f \) such that

\[
\phi_f(x_1, \ldots, x_k, y)
\]
is true if and only if

\[
y = f(x_1, \ldots, x_k)
\]

First incompleteness Theorem (Gödel)
That theory is not recursively axiomatizable

Non-linear Integer Arithmetic

(a.k.a. Peano Arithmetic)

First-Order Integer Arithmetic
All valid first-order formulas on integers with +, × and ≤

- This theory is not even semi-decidable
- SMT solvers implement incomplete satisfiability checks: if solver answers 'unsat' then it is unsatisfiable

Demo with Why3
arith.mlw

Summary of prover limitations

- Superposition solvers (E, Spass, )
  - do not support well theories except equality
  - do quite well with quantifiers
- SMT solvers (Alt-Ergo, CVC4, Z3)
  - several theories are built-in
  - weaker with quantifiers
- None support reasoning by induction
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Exercises

IMP language

A very basic imperative programming language
- only global variables
- only integer values for expressions
- basic statements:
  - assignment $x \leftarrow e$
  - sequence $S_1; S_2$
  - conditionals if $e$ then $S_1$ else $S_2$
  - loops while $e$ do $S$
  - no-op skip

Formal Contracts

General form of a program:

Contract
- **precondition**: expresses what is assumed before running the program
- **post-condition**: expresses what is supposed to hold when program exits

Demo with Why3
contracts.mlw

Hoare triples

- **Hoare triple** : notation $\{P\}s\{Q\}$
- $P$ : formula called the precondition
- $Q$ : formula called the postcondition

Intended meaning
$\{P\}s\{Q\}$ is true if and only if:
when the program $s$ is executed in any state satisfying $P$, then
(if execution terminates) its resulting state satisfies $Q$

This is a **Partial Correctness**: we say nothing if $s$ does not terminate
Examples

Examples of valid triples for partial correctness:
- \( \{ x = 1 \} x \leftarrow x + 2 \{ x = 3 \} \)
- \( \{ x = y \} x \leftarrow x + y \{ x = 2y \} \)
- \( \{ \exists v. x = 4 \cdot v \} x \leftarrow x + 42 \{ \exists w. x = 2 \cdot w \} \)
- \( \{ true \} \text{while} 1 \text{do} \text{skip} \{ false \} \)

Running Example

Three global variables \( n, \text{count}, \) and \( \text{sum} \)

\[
\begin{align*}
\text{count} & \leftarrow 0; \text{sum} \leftarrow 1; \\
\text{while} \text{sum} \leq n \text{ do} \\
& \quad \text{count} \leftarrow \text{count} + 1; \text{sum} \leftarrow \text{sum} + 2 \cdot \text{count} + 1
\end{align*}
\]

What does this program compute?

(assuming input is \( n \) and output is \( \text{count} \))

Informal specification:
- at the end of execution of this program, \( \text{count} \) contains the square root of \( n \), rounded downward
- e.g. for \( n=42 \), the final value of \( \text{count} \) is 6.

See file \texttt{imp_isqrt.mlw}

Hoare logic as an Axiomatic Semantics

Original Hoare logic \([\sim 1970]\)

\textbf{Axiomatic Semantics} of programs

Set of inference rules producing triples

\[
\begin{align*}
\{ P \} \text{skip} \{ P \} \\
\{ P \} x \leftarrow e \{ P \} x \leftarrow e \{ P \} \\
\{ P \} s_1 \{ Q \} \quad \{ Q \} s_2 \{ R \} \\
\{ P \} s_1 ; s_2 \{ R \}
\end{align*}
\]

- Notation \( P[x \leftarrow e] \): replace all occurrences of program variable \( x \) by \( e \) in \( P \).

Hoare Logic, continued

Frame rule:

\[
\frac{\{ P \} \{ Q \}}{\{ P \land R \} \{ Q \land R \}}
\]

with \( R \) a formula where no variables assigned in \( s \) occur

Consequence rule:

\[
\frac{\{ P' \} \{ Q' \} \models P \rightarrow P' \quad Q' \rightarrow Q}{\{ P \} \{ Q \}}
\]

- Example: proof of

\[
\{ x = 1 \} x \leftarrow x + 2 \{ x = 3 \}
\]
Hoare Logic, continued

Rules for if and while:

\[
\begin{align*}
\{ P \land e \} s_1 \{ Q \} & \quad \{ P \land \neg e \} s_2 \{ Q \} \\
\{ P \} & \text{if } e \text{ then } s_1 \text{ else } s_2 \{ Q \} \\
\{ I \land e \} s \{ I \} & \quad \{ I \} \text{while } e \text{ do } s \{ I \land \neg e \}
\end{align*}
\]

\( I \) is called a loop invariant.

Example: \( \text{isqrt}(42) \)

Exercise: prove of the triple

\[
\{ n \geq 0 \} \text{ISQRT} \{ \text{count}^2 \leq n \land n < (\text{count} + 1)^2 \}
\]

Could we do that by hand?

Back to demo: file \texttt{imp_isqrt.mlw}

**Warning**
Finding an adequate loop invariant is a major difficulty

Beyond Axiomatic Semantics

- Operational Semantics
- Semantic Validity of Hoare Triples
- Hoare logic as correct deduction rules

Operational semantics

*[Plotkin 1981, structural operational semantics (SOS)]*

- we use a standard small-step semantics
- **program state**: describes content of global variables at a given time. It is a finite map \( \Sigma \) associating to each variable \( x \) its current value denoted \( \Sigma(x) \).
- **Value of an expression** \( e \) in some state \( \Sigma \):
  - denoted \( [e]_\Sigma \)
  - always defined, by the following recursive equations:
    \[
    \begin{align*}
    [n]_\Sigma &= n \\
    [x]_\Sigma &= \Sigma(x) \\
    [e_1 \ op \ e_2]_\Sigma &= [e_1]_\Sigma \ [op] \ [e_2]_\Sigma
    \end{align*}
    \]
  - \([op] \) natural semantic of operator \( op \) on integers (with relational operators returning 0 for false and \( \neq \) 0 for true).
Semantics of statements

Semantics of statements: defined by judgment

\[ \Sigma, s \leadsto \Sigma', s' \]

meaning: in state \( \Sigma \), executing one step of statement \( s \) leads to the state \( \Sigma' \) and the remaining statement to execute is \( s' \).

The semantics is defined by the following rules.

\[ \Sigma, x \leftarrow e \leadsto \Sigma \{ x \leftarrow [e]_\Sigma \}, \text{skip} \]

\[ \Sigma, s_1 \leadsto \Sigma', s'_1 \]

\[ \Sigma, (s_1; s_2) \leadsto \Sigma', (s'_1; s_2) \]

\[ \Sigma, (\text{skip}; s) \leadsto \Sigma, s \]

Semantics of statements, continued

\[ [e]_\Sigma \neq 0 \]

\[ \Sigma, \text{if } e \text{ then } s_1 \text{ else } s_2 \leadsto \Sigma, s_1 \]

\[ [e]_\Sigma = 0 \]

\[ \Sigma, \text{if } e \text{ then } s_1 \text{ else } s_2 \leadsto \Sigma, s_2 \]

\[ [e]_\Sigma \neq 0 \]

\[ \Sigma, \text{while } e \text{ do } s \leadsto \Sigma, (s; \text{while } e \text{ do } s) \]

\[ [e]_\Sigma = 0 \]

\[ \Sigma, \text{while } e \text{ do } s \leadsto \Sigma, \text{skip} \]

Execution of programs

▷ \( \leadsto \): a binary relation over pairs (state,statement)

▷ transitive closure: \( \leadsto^+ \)

▷ reflexive-transitive closure: \( \leadsto^* \)

In other words:

\[ \Sigma, s \leadsto^* \Sigma', s' \]

means that statement \( s \), in state \( \Sigma \), reaches state \( \Sigma' \) with remaining statement \( s' \) after executing some finite number of steps.

Running example:

\[ \{ n = 42, count = ?, sum = ? \}, \text{ISQRT} \leadsto^* \]

\[ \{ n = 42, count = 6, sum = 49 \}, \text{skip} \]

Execution and termination

▷ any statement except \( \text{skip} \) can execute in any state

▷ the statement \( \text{skip} \) alone represents the final step of execution of a program

▷ there is no possible runtime error.

Definition

Execution of statement \( s \) in state \( \Sigma \) **terminates** if there is a state \( \Sigma' \) such that \( \Sigma, s \leadsto^* \Sigma', \text{skip} \)

▷ since there are no possible runtime errors, \( s \) does not terminate means that \( s \) **diverges** (i.e. executes infinitely).
Semantics of formulas

\([p]_\Sigma : \)
- semantics of formula \(p\) in program state \(\Sigma\)
- is a logic formula where no program variables appear anymore
- defined recursively as follows.

\[\begin{align*}
\left[\text{e}\right]_\Sigma &= \left[\text{e}\right]_\Sigma \neq 0 \\
\left[p_1 \land p_2\right]_\Sigma &= \left[p_1\right]_\Sigma \land \left[p_2\right]_\Sigma \\
&\ldots \\
\end{align*}\]

where semantics of expressions is augmented with

\[\begin{align*}
\left[v\right]_\Sigma &= v \\
\left[x\right]_\Sigma &= \Sigma(x)
\end{align*}\]

Notations:
- \(\Sigma \models p\): the formula \([p]_\Sigma\) is valid
- \(\models p\): formula \([p]_\Sigma\) holds in all states \(\Sigma\).

Soundness

Definition (Partial correctness)
Hoare triple \(\{P\} s\{Q\}\) is said valid if:
for any states \(\Sigma, \Sigma',\) if
- \(\Sigma, s \rightarrow^* \Sigma',\) skip and
- \(\Sigma \models P\)
then \(\Sigma' \models Q\)

Theorem (Soundness of Hoare logic)
The set of rules is correct: any derivable triple is valid.

This is proved by induction on the derivation tree of the considered triple.
For each rule: assuming that the triples in premises are valid, we show that the triple in conclusion is valid too.

Annotated Programs

Goal
Add automation to the Hoare logic approach
We augment IMP with explicit loop invariants

while e invariant I do s
Weakest liberal precondition

[Dijkstra 1975]

Function $WLP(s, Q)$:

- $s$ is a statement
- $Q$ is a formula
- returns a formula

It should return the minimal precondition $P$ that validates the triple $\{P\}s\{Q\}$

Definition of $WLP(s, Q)$

Recursive definition:

\[
\begin{align*}
WLP(\text{skip}, Q) &= Q \\
WLP(x < e, Q) &= Q[x \leftarrow e] \\
WLP(s_1; s_2, Q) &= WLP(s_1, WLP(s_2, Q)) \\
WLP(\text{if } e \text{ then } s_1 \text{ else } s_2, Q) &= (e \rightarrow WLP(s_1, Q)) \land (\neg e \rightarrow WLP(s_2, Q))
\end{align*}
\]

Examples

\[
\begin{align*}
WLP(x < x + y, x = 2y) &\equiv x + y = 2y \\
WLP(\text{while } y > 0 \text{ invariant even}(y) \text{ do } y < y - 2, \text{even}(y)) &\equiv \text{even}(y) \land \\
&\forall v, ((v > 0 \land \text{even}(v)) \rightarrow \text{even}(v - 2)) \\
&\land ((v \leq 0 \land \text{even}(v)) \rightarrow \text{even}(v))
\end{align*}
\]
Soundness

Theorem (Soundness)
For all statement $s$ and formula $Q$, $\{WLP(s, Q)\}s \{Q\}$ is valid.

Proof by induction on the structure of statement $s$.

Consequence
For proving that a triple $\{P\}s\{Q\}$ is valid, it suffices to prove the formula $P \rightarrow WLP(s, Q)$.

This is basically the goal that Why3 produces.

Digression: Completeness of Hoare Logic

Two major difficulties for proving a program
- guess the appropriate intermediate formulas (for sequence, for the loop invariant)
- prove the logical premises of consequence rule

Theoretical question: completeness. Are all valid triples derivable from the rules?

Theorem (Relative Completeness of Hoare logic)
The set of rules of Hoare logic is relatively complete: if the logic language is expressive enough, then any valid triple $\{P\}s\{Q\}$ can be derived using the rules.

[Cook, 1978]“Expressive enough”: representability of any recursive function
Yet, this does not provide an effective recipe to discover suitable loop invariants (see also the theory of abstract interpretation [Cousot, 1990])

Exercise 1

Consider the following (inefficient) program for computing the sum $a + b$.

```plaintext
x <- a; y <- b;
while y > 0 do
  x <- x + 1; y <- y - 1
```

(Why3 file to fill in: imp_sum.mlw)

- Propose a post-condition stating that the final value of $x$ is the sum of the values of $a$ and $b$
- Find an appropriate loop invariant
- Prove the program.
Exercise 2

The following program is one of the original examples of Floyd.

\[
q \leftarrow 0; \ r \leftarrow x;
\]
\[
\textbf{while } r \geq y \textbf{ do}
\]
\[
r \leftarrow r - y; \ q \leftarrow q + 1
\]
(Why3 file to fill in: imp_euclide.mlw)

- Propose a formal precondition to express that \(x\) is assumed non-negative, \(y\) is assumed positive, and a formal post-condition expressing that \(q\) and \(r\) are respectively the quotient and the remainder of the Euclidean division of \(x\) by \(y\).
- Find appropriate loop invariant and prove the correctness of the program.

Exercise 3

Let's assume given in the underlying logic the functions \(\text{div2}(x)\) and \(\text{mod2}(x)\) which respectively return the division of \(x\) by 2 and its remainder. The following program is supposed to compute, in variable \(r\), the power \(x^n\).

\[
r \leq 1; \ p \leftarrow x; \ e \leftarrow n;
\]
\[
\textbf{while } e > 0 \textbf{ do}
\]
\[
\text{if } \text{mod2}(e) \neq 0 \text{ then } r \leftarrow r * p;
\]
\[
p \leftarrow p * p;
\]
\[
e \leftarrow \text{div2}(e);
\]
(Why3 file to fill in: power_int.mlw)

- Assuming that the power function exists in the logic, specify appropriate pre- and post-conditions for this program.
- Find an appropriate loop invariant, and prove the program.

Exercise 4

The Fibonacci sequence is defined recursively by \(\text{fib}(0) = 0\), \(\text{fib}(1) = 1\) and \(\text{fib}(n + 2) = \text{fib}(n + 1) + \text{fib}(n)\). The following program is supposed to compute \(\text{fib}\) in linear time, the result being stored in \(y\).

\[
y \leftarrow 0; \ x \leftarrow 1; \ i \leftarrow 0;
\]
\[
\textbf{while } i < n \textbf{ do}
\]
\[
aux \leftarrow y; \ y \leftarrow x; \ x \leftarrow x + aux; \ i \leftarrow i + 1
\]

- Assuming \(\text{fib}\) exists in the logic, specify appropriate pre- and post-conditions.
- Prove the program.

Exercise (Exam 2011-2012)

In this exercise, we consider the simple language of the first lecture of this course, where expressions do not have any side effect.

1. Prove that the triple

\[
\{ \mathbf{P} \} \ x \gets e \{ \exists v, e[x \leftarrow v] = x \land \mathbf{P}[x \leftarrow v] \}
\]

is valid with respect to the operational semantics.

2. Show that the triple above can be proved using the rules of Hoare logic.

Let us assume that we replace the standard Hoare rule for assignment by the rule

\[
\{ \mathbf{P} \} \ x \gets e \{ \exists v, e[x \leftarrow v] = x \land \mathbf{P}[x \leftarrow v] \}
\]

3. Show that the triple \(\{ \mathbf{P}[x \leftarrow e]\} x \gets e(\mathbf{P})\) can be proved with the new set of rules.


